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Diamagnetic properties of metamaterials: a magnetostatic analogy

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Abstract. The response of a metamaterial, consisting of a 3D lattice of lossy capacitively loaded metallic loops is studied theoretically when it is inserted into a homogeneous harmonically varying magnetic field. The current distribution is found by taking into account the magnetic coupling between any pair of loops in the approximation of no retardation. It is shown that in a frequency range above its resonant frequency the metamaterial behaves as a diamagnet expelling the applied magnetic field. As the resonant frequency is approached the magnetic field is shown to be expelled not only from the volume of the metamaterial but from a larger zone which in the vicinity of the resonant frequency takes the form of a sphere. In the lossless case the radius of this exclusion sphere tends to infinity. In the presence of losses the maximum radius is limited by the quality factor of the individual elements. The response of a single element is shown to be analogous to that of a sphere of magnetic material, an analogy that leads to an alternative definition of effective permeability.

PACS. 41.20.-q Applied classical electromagnetism – 75.20.-g Diamagnetism, paramagnetism, and superparamagnetism – 75.40.Cx Static properties (order parameter, static susceptibility, heat capacities, critical exponents, etc.) – 75.40.Gb Dynamic properties (dynamic susceptibility, spin waves, spin diffusion, dynamic scaling, etc.)

1 Introduction

The term metamaterial has only recently been coined. It means that the properties (meaning essentially electromagnetic properties) of a particular material in a narrow frequency range can be radically changed by the periodic or random inclusion of small elements which have one or more resonances within the required band. By small it is meant that the dimensions of the element are small relative to the wavelength and the element may also be small relative to the unit cell. The basic idea of including metallic spheres [1], discs into a dielectric has been around for at least half a century. The aim at the time was to change the dielectric constant. The results achieved were modest, interest soon waned in the subject. However if we look at metamaterials as the descendants of periodic structures then of course their previous history takes many volumes, starting with X-ray diffraction by crystals and diffraction gratings and ending up with a large variety of engineering devices like artificial delay lines, slow wave structures in microwave tubes, phased-array antennas, Bragg reflection mirrors in lasers, etc. So if this kind of devices have been known for such a long time why was there a need to coin a new term? The need arose due to predictions of entirely new phenomena which come about when negative permeability and negative permittivity can be simultaneously realized in the same frequency band. The seminal paper was written in 1968 by Veselago [2], largely forgotten until recently revived by Smith et al. [3] and Pendry [4].

The concept of negative permittivity has often been used to describe plasmas, whether in vacuum or in a metal, below the plasma frequency. The physics is completely understood and the mathematical treatment, at least for describing basic phenomena, is quite straightforward. The situation has been different concerning negative permeability. It has been shown to be possible [5,6], but a relatively simple realization came only five years ago, in a paper by Pendry et al. [7].

By now negative permeability has been well established [3,8] although there is no general consensus as yet concerning its measurement nor is there an agreed recipe how to calculate the effective magnetic permeability. The definition by Gorkunov et al. [9] for example differs from that of Pendry et al. [7].

Our aim in this paper is to approach the problem of effective permeability from a different angle, from that

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of the diamagnetic properties of magnetic materials. The relative permeability of diamagnets is known to vary between 1 and 0. It is zero when the magnetic flux is completely excluded. In a metamaterial, as we shall show, the magnetic flux can be excluded from a volume larger than the material itself, a phenomenon that can be referred to as superdiamagnetism and which can also be interpreted as due to the material having negative permeability.

In Section 2 we shall describe our mathematical model for a 3D lattice of capacitively loaded loops. In Section 3 we shall investigate the effect of a single element and of a 3D metamaterial when immersed in a homogeneous harmonically varying magnetic field. A magnetostatic analogy is discussed in Section 4 and conclusions are drawn in Section 5.

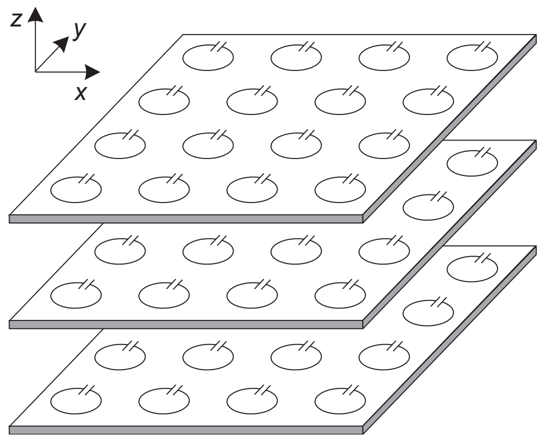

Fig. 1. A 3D lattice of capacitively loaded loops.

2 Mathematical model

The metamaterial element chosen for its magnetic properties is a capacitively loaded metallic loop (Fig. 1). The capacitance is represented by parallel plates although in practice [10] it would take the form of a capacitor soldered between the ends of the loop. The radius of the loop is denoted by r_0 and that of the wire by r_{w}. The loop impedance may be taken as that of a series LCR circuit

$$Z_0 = j\left(\omega L - \frac{1}{\omega C}\right) + R, \tag{1}$$

where L is the self-inductance, C is the capacitance, R is the resistance and ω is the frequency of the varying magnetic field. The resonant frequency of the circuit is $\omega_0 = 1/\sqrt{LC}$.

Our metamaterial is assumed in the form of N identical elements making up a regular 3D lattice in which the distance between the elements is much smaller than the wavelength. We shall investigate the properties of this structure when inserted into a homogeneous magnetic field of amplitude H_0 taken perpendicular to the plane of the loops. Then each of the elements is excited by the same voltage $V = -j\omega H_0 \pi r_0^2$ due to the variation of the magnetic flux across the plane of the loop. I_n, the current in loop n, will induce a voltage $Z_{m,n}I_n$ in loop m, where $Z_{m,n} = Z_{n,m}$ is the mutual impedance between loops m and n. It is related to the mutual inductance between two loops $M_{m,n} = M_{n,m}$ as $Z_{m,n} = j\omega M_{m,n}$.

The current in the mth loop is related to the other currents by the relationship [11]

$$Z_0 I_m + \sum_{n\neq m} Z_{m,n} I_n = V. \tag{2}$$

The general relationship may also be written in matrix form,

$$\mathbf{V} = Z\mathbf{I}, \tag{3}$$

where $\mathbf{V}$ and $\mathbf{I}$ are N-dimensional vectors

$$\mathbf{V} = (V_1, V_2, ..., V_m, ...V_N)\,; \qquad \mathbf{I} = (I_1, I_2, ..., I_m, ...I_N) \tag{4}$$

and Z is a symmetrical $N \times N$ matrix with non-diagonal elements $Z_{m,n}$ and diagonal elements Z_0. Equation (3) is nothing else but the generalized Ohm's law.

Knowing the current in each loop, the magnetic field due to each current may be obtained in terms of elliptic functions [12]. Total induced magnetic field is then obtained by superimposing the effects of all currents.

Before we can start the calculations we need to decide on the parameters. We shall take $\omega_0/(2\pi) = 63.87$ MHz, the frequency of nuclear magnetic resonance at 1.5 tesla, since one of the potential applications of metamaterials is in Magnetic Resonance Imaging [13]. We shall further assume that the loops of radius $r_0 = 10$ mm are made of copper wires having a diameter of $2\,r_w = 2$ mm. The self-inductance may then be determined from the well known expression given by most books on the subject of electromagnetism (see e.g. [14]). The value of the capacitance then follows from our choice of the resonant frequency, ω_0. The expression for the mutual inductance between two loops is also available in the literature [12]. In the approximation that all dimensions are small relative to the wavelength and retardation is negligible it may be obtained by one contour integration where the integrand is an elliptic function. The resistance is calculated from the given dimensions of the loop for a conductivity of 5.8×10^7 S/m taking the skin effect into account. At our chosen frequency the circuit parameters are then, $L = 33$ nH, $C = 187$ pF and $R = 20.5$ mΩ where the capacitor is assumed to be lossless.

3 Results

3.1 Single loop

We shall start with the simplest case of a single lossless loop in the x-y plane inserted into a harmonically varying magnetic field directed along the z axis. In the absence of the capacitor the current in the loop will create a magnetic field opposing the applied field. A single loop is clearly a diamagnet. Now let us put back the capacitor. At a frequency at which the impedance is capacitive (i.e. below

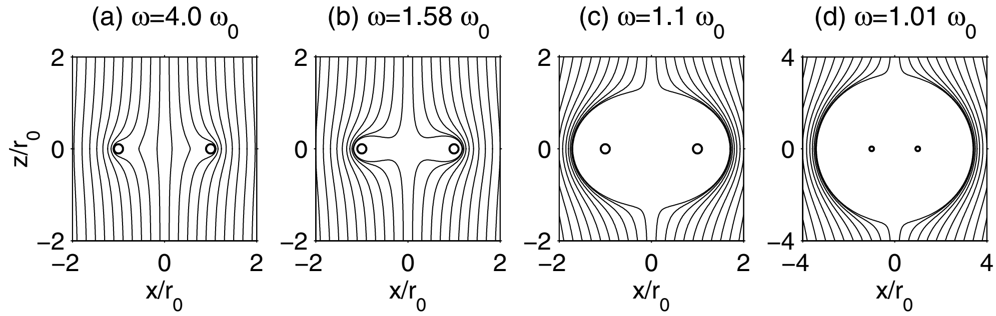

Fig. 2. Single loop. Streamlines of the total magnetic field for $\omega/\omega_0 = 2$, 1.58, 1.1, 1.01 (a–d).

the resonant frequency) the current induced by the magnetic field will reinforce the applied magnetic field. When the frequency is above the resonant frequency then the loaded loop will again have an inductive impedance but the magnitude of the current will now depend on how close the frequency is to the resonant one. So in the vicinity of the resonant frequency (but still above it) the current will be large, therefore a larger diamagnetic effect can be expected.

Let us now see the curves calculated from our theoretical model for a single capacitively loaded loop of resonant frequency ω_0. Figures 2a–d show how the external field is increasingly diverted away from the loop as the frequency decreases from 4 ω_0 to 1.01 ω_0 where the zone of exclusion may be seen to extend to several times the loop radius. By exclusion we mean here that the original magnetic field cannot penetrate that zone. It does not mean however that there is no magnetic field inside the zone. Figures 3a and b show both the internal and external fields for $\omega = 1.1\,\omega_0$ and $1.01\,\omega_0$. As the frequency gets nearer the resonant frequency the exclusion zone takes the form of a sphere. The physical picture that springs to mind is that the internal field produced by the current acts as a kind of airbag which prevents the penetration of the external flux. Our numerical calculations show that as the frequency approaches the resonant frequency the radius of the exclusion sphere, r_{e}, increases very fast. An approximate analytical expression derived in the Appendix does indeed suggest that $r_{\mathrm{e}} \to \infty$ as $\omega \to \omega_0$.

Let us now include losses. As may be expected the size of the exclusion zone will then be limited. According to equation (10) of the Appendix the maximum radius will be proportional to $Q^{1/3}$ where Q, the quality factor, is equal to $\omega L/R$. This is a result that we could expect on physical grounds. The quality factor is related to stored energy which may be taken to be proportional to the volume of the exclusion sphere.

The radius of the exclusion sphere at the optimum frequency may also be obtained from numerical calculations as demonstrated in Figures 4a–c. As Q declines by factors of 8 from 6400 to 800 to 100 the radius of the sphere may be seen to decrease by the cubic root of 8. We have to note that, strictly speaking, it is no longer possible to draw flux lines in the presence of losses because the magnetic field becomes elliptically polarized so the magnetic field has not got a definite direction in a given point in space. However for a large enough Q the geometrical representation would still give good approximation.

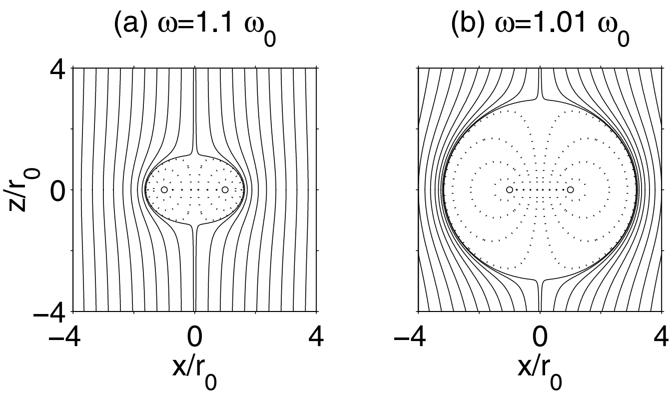

Fig. 3. Single loop. Streamlines of the total magnetic field for $\omega/\omega_0 = 1.1$ (a), 1.01 (b). Internal field is shown by dotted lines.

3.2 3D lattice of loops

It has been convenient to investigate the single loop first and draw some conclusions. We wish to show now that similar conclusions apply when we take a chunk of metamaterial consisting of a 3D lattice of 3 by 3 by 8 loops. The distance between the nearest loops is taken as $a = 2.25\,r_0$ in the horizontal plane and $b = 0.25\,r_0$ in the vertical direction. For simplicity we shall neglect losses again. It may be seen (Fig. 5) that varying the frequency the lattice of loops shows the same behaviour as the single loop. At high frequency $\omega = 2\,\omega_0$ the flux can penetrate the structure which can be seen to be slightly diamagnetic (Fig. 5a). As we reduce the frequency, the region from which the external flux is excluded becomes again larger. At $\omega = 0.78\,\omega_0$ the flux leakage through the structure is stopped (Fig. 5b) and at $\omega = 0.67\,\omega_0$ the exclusion region resembles a sphere again (Fig. 5c). The radius may be seen to be large for $\omega = 0.65\omega_0$ (Fig. 5d) and it tends to infinity as we approach the resonant frequency. The main difference from the single loop case is that the frequency at which the exclusion sphere tends to infinity is now well below the resonant frequency of the single loop due to the

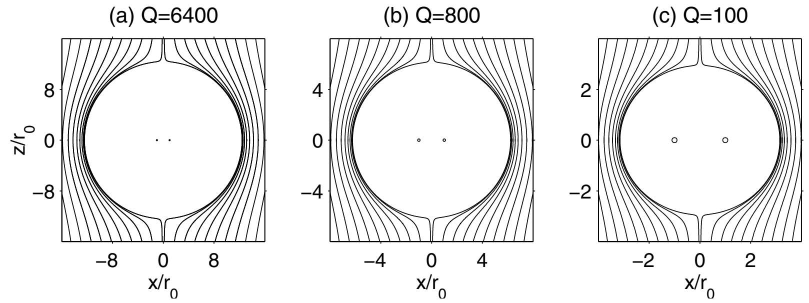

Fig. 4. Single loop. Streamlines of the total magnetic field in the presence of losses for $Q = 6400$, 800, 100 (a–c).

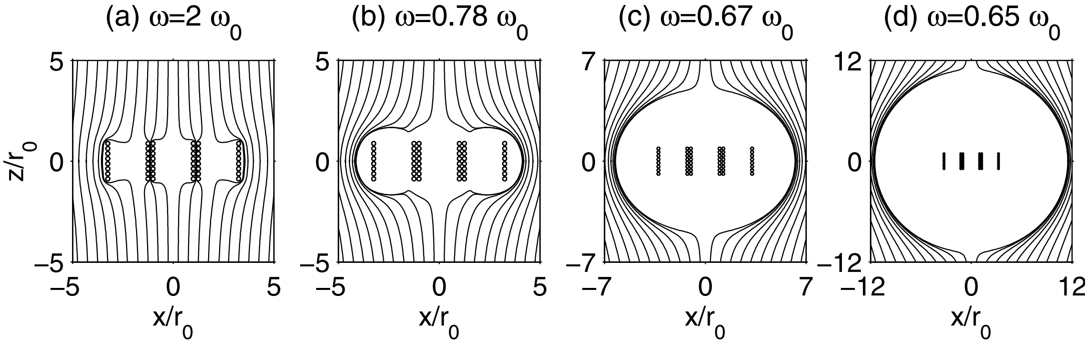

Fig. 5. 3D lattice. Streamlines of the total magnetic field for $\omega/\omega_0 = 2$, 0.78, 0.67, 0.65 (a–d).

interactions between the elements. Losses will of course limit again the size of the exclusion region.

Varying the density or the shape of the metamaterial will also influence the results. The coupling between two loops on the same vertical axis is positive, the coupling between two loops in the same horizontal plane is negative, the coupling between loops shifted both vertically and horizontally with respect to each other is quite close to zero. By varying the coupling strength between the loops (e.g. by varying the lattice constant or the lattice configuration) will lead to a shift of the resonant frequency approaching which the metamaterial expels the external flux more and more effectively.

4 Magnetostatic analogy: sphere of permeability μ_r in an external field

It is well known, it may be found in most textbooks on electromagnetic theory (see e.g. [14]), that a sphere of magnetic material will partially exclude an externally applied static magnetic field when the relative permeability is in the range $0 < \mu_r < 1$. When $\mu_r = 1$ the magnetic field is unaffected. When $\mu_r = 0.2$ a considerable part of the magnetic field is expelled (Fig. 6a) and at $\mu_r = 0$ the sphere is an ideal diamagnet, the magnetic field is completely expelled (Fig. 6b). The corresponding magnetic fields inside and outside of the sphere are given by the expressions (see e.g. [12])

$$B_{\mathrm{in,z}} = \mu_0\, H_0 \frac{3\mu_r}{\mu_r + 2}$$

$$B_{\mathrm{out,r}} = \mu_0\, H_0 \cos\theta \left[1 + 2\left(\frac{R_0}{r}\right)^3 \frac{\mu_r - 1}{\mu_r + 2}\right]$$

$$B_{\mathrm{out},\theta} = \mu_0\, H_0 \sin\theta \left[-1 + \left(\frac{R_0}{r}\right)^3 \frac{\mu_r - 1}{\mu_r + 2}\right], \quad (5)$$

where μ_0 is the permeability of the vacuum, $\mu_0\, H_0$ is the originally applied magnetic flux density, R_0 is the radius of the sphere, r, θ (θ measured from the z axis) are spherical coordinates with the origin of the coordinate system being located at the centre of the sphere.

As far as we know the above expressions from which Figures 6a, b have been plotted had only been used for μ_r positive or zero. We shall now enter uncharted territory and assume that negative μ_r is also feasible and see what the expressions will lead to. Taking $\mu_r = -1$ the sphere of magnetic material expels the external magnetic field not only from its own volume but from a larger one as may be seen in Figure 6c. As we further reduce μ_r to -1.8 the exclusion sphere becomes even larger (Fig. 6d). The radius of the exclusion sphere can be calculated from equations (5) to yield

$$r_{\mathrm{e}}^3 = 2\,\frac{1 - \mu_r}{\mu_r + 2} R_0^3 \quad (6)$$

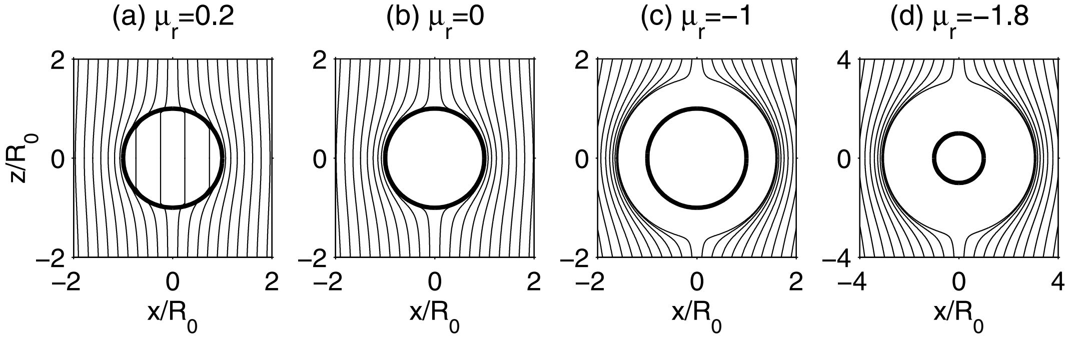

Fig. 6. A sphere of permeability μ_r in a magnetostatic field. Streamlines of the total magnetic field for $\mu_r = 0.2$, 0, −1, −1.8 (a–d).

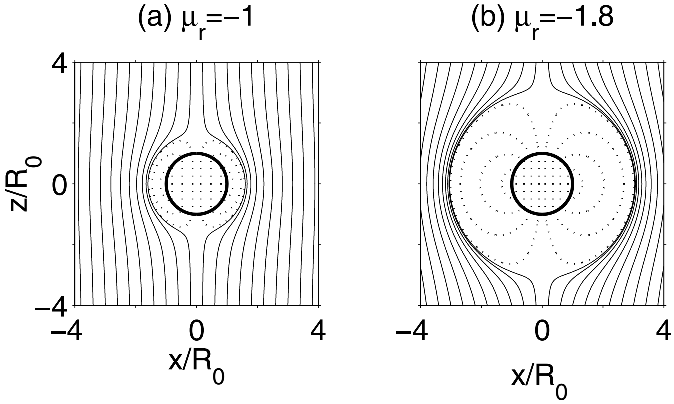

Fig. 7. A sphere of permeability μ_r in a magnetostatic field. Streamlines of the total magnetic field for $\mu_r = -1$ (a), −1.8 (b). Internal field is shown by dotted lines.

and can be seen to become infinitely large at $\mu_r = -2$. Interestingly, the pattern of the internal magnetic field (dotted lines in Figs. 7a, b) resembles very much that of our capacitively loaded metamaterial (compare Figs. 7a, b with Figs. 3a, b).

There is clearly a formal analogy between the response of our metamaterial and that of a magnetic sphere to an applied homogeneous magnetic field. It is then difficult to resist the temptation to assign a frequency dependent effective permeability, based on this analogy, to the metamaterial.

We could associate with a single loop negative permeability between 0 and −2 as the radius of the exclusion sphere increases. The analogy is less perfect for the 3D lattice but as Figure 5 shows for a particular example the exclusion sphere may also tend to infinity as the frequency ω_0 is approached. Hence, again, we could say that the infinite exclusion sphere corresponds to an effective permeability of −2.

There is no doubt that an effective permeability may be defined, as it was done in the past, by the effect of the metamaterial upon a transverse electromagnetic wave. But that is not the only possible definition. We claim that the definition should, or could, depend on the experimental arrangement. If the metamaterial is immersed in a homogeneous magnetic field and we are interested in its diamagnetic properties then the definition advanced in this paper is an equally plausible one.

5 Conclusions

Our principal results show that a lossless metamaterial can not only expel an external magnetic field (thus behaving as a diamagnet) from the volume it occupies but, in principle, it can expel an external magnetic field from an arbitrarily large sphere. In the presence of losses the radius of the sphere has been shown to be limited by the quality factor of the elements. A formal analogy has also been presented between the diamagnetic properties of the metamaterial and that of a sphere of magnetic material in a static magnetic field and this analogy has been used for a definition of effective permeability.

All the analyses have been done in the frequency range above the resonant frequency (whether that of a single loop or of a metamaterial) which leads to diamagnetic properties. A similar study below the resonant frequency would show paramagnetic properties, the possibility of concentrating an externally applied field where the amount of concentration is limited only by the quality factor of the elements. We hope to return to that problem in a future publication.

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Appendix: Derivation of approximate analytical expressions for the radius of the exclusion sphere due to a single loop

Assuming that flux exclusion takes place far away from the loop (i.e. the radius of the exclusion sphere is large relative to the loop radius) we may find the magnetic field from the well known expression (see e.g. [14]) for a magnetic dipole. We shall need only the radial component which is given by

$$H_r = \frac{I\, r_0^2}{2\, r^3}\cos\theta \qquad H_\theta = \frac{I\, r_0^2}{4\, r^3}\sin\theta, \tag{7}$$

where the angle θ is measured from the z axis. We shall now argue that the radius of the sphere is determined by the point where the dipole's radial field just cancels the applied field. This happens at $\theta = 0$ when $H_r = H_0$. For the lossless case the radius may then be obtained in the form

$$r_\mathrm{e}^3 = \frac{\pi}{2}\, r_0^3\, \frac{\mu_0\, r_0}{L}\, \frac{\omega^2}{\omega^2 - \omega_0^2}. \tag{8}$$

Another way of determining r_e is to argue that the flux that is expelled from the exclusion sphere must appear in the plane $\theta = \pi/2$ as extra flux outside the exclusion sphere. In other words r_e can be calculated from the equation

$$\mu_0\, H_0\, \pi\, r_\mathrm{e}^2 = 2\pi\, \mu_0 \int_{r_\mathrm{e}}^{\infty} \mu_0\, H_\theta \left(\theta = \frac{\pi}{2}\right) r\, dr. \tag{9}$$

It turns out that this latter definition leads to the same expression as equation (8).

In the presence of losses the magnetic field will have elliptic polarization, i.e. it is no longer possible to give the magnetic field a definite direction in space. It would be still possible to define r_e by the criterion that has led to equation (9) for all values of $Q\, (= \omega L/R)$ and ω and find numerically the frequency that maximizes the exclusion zone. Since these calculations are rather laborious and lead to no additional physical insight we shall not do them here.

We can however obtain the optimum frequency from a simpler but rather rough argument that stipulates that it is sufficient if H_0 cancels the real part of H_r at $\theta = 0$ (i.e. to ignore the imaginary component at that point). This will lead to the simple solution

$$r_{\mathrm{e,max}}^3 = \frac{\pi}{4}\, r_0^3\, \frac{\mu_0\, r_0}{L}\, Q. \tag{10}$$

References

1. R.E. Collin, *Field theory of guided waves* (Oxford University Press, Oxford, 1991)
2. V.G. Veselago, Sov. Phys. Usp. **10**, 509 (1968)
3. D.R. Smith, W.J. Padilla, D.C. Vier, S.C. Nemat-Nasser, S. Schultz, Phys. Rev. Lett. **84**, 4184 (2000)
4. J.B. Pendry, Phys. Rev. Lett. **85**, 3966 (2000)
5. D.L. Mills, E. Burstein, Rep. Prog. Phys. **37**, 817 (1974)
6. R.E. Camley, D.L. Mills, Phys. Rev. B **26**, 1280 (1982)
7. J.B. Pendry, A.J. Holden, D.J. Robbins, W.J. Stewart, IEEE Trans. MTT **47**, 2075 (1999)
8. R.A. Shelby, D.R. Smith, S. Schultz, Science **292**, 77 (2001)
9. M. Gorkunov, M. Lapine, E. Shamonina, K.H. Ringhofer, Eur. Phys. J. B **28**, 263 (2002)
10. M.C.K. Wiltshire, E. Shamonina, I.R. Young, L. Solymar, Electron. Lett. **39**, 215 (2003)
11. E. Shamonina, V.A. Kalinin, K.H. Ringhofer, L. Solymar, J. Appl. Phys. **92**, 6252 (2002)
12. L.D. Landau, E.M. Lifshitz, *Electrodynamics of continuous media* (Pergamon Press, Oxford, 1984)
13. M.C.K. Wiltshire, J.B. Pendry, I.R. Young, D.J. Larkman, D.J. Gilderdale, J.V. Hajnal, Science **291**, 849 (2001)
14. L. Solymar, *Lectures on electromagnetic theory* (Oxford University Press, Oxford, 1984)